# Kazhdan's property (T) for $\mathrm{Aut}(\mathbf{F}_n)$ and $\mathrm{EL}_n(\mathcal{R})$

Narutaka OZAWA (小澤 登高)

RIMS, Kyoto University

PRIMA 2022, Vancouver, December 07

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C*-algebras are an esoteric subject —"the most abstract nonsense that exists in mathematics," in Casazza's words. "Nobody outside the area knows much about it."

Quanta Magazine: *'Outsiders' Crack 50-Year-Old Math Problem.*

http://www.quantamagazine.org/computer-scientists-solve-kadison-singer-problem-20151124

**Theorem (Kazhdan 1967)**

Any simple Lie group $G$ of real rank $\geq 2$ (e.g., $G = \mathrm{SL}_n(\mathbb{R})$, $n \geq 3$) and its lattice $\Gamma$ (e.g., $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$) have **property (T)**.

$\rightsquigarrow$ $\Gamma$ is finitely generated and has finite abelianization.

Throughout this talk, $\Gamma = \langle S \rangle$ is a finitely generated group.

**Definition (for discrete groups)**

$\Gamma$ has (T) $\overset{\text{def}}{\iff}$ $\exists \kappa = \kappa(\Gamma, S) > 0$ s.t. $\forall (\pi, \mathcal{H})$ unitary rep'n and $\forall v \in \mathcal{H}$

$$d(v, \mathcal{H}^\Gamma) \leq \kappa^{-1} \max_{s \in S} \|v - \pi(s)v\|,$$

i.e., an almost invariant vector $v$ is close to an invariant vector $\mathrm{Proj}_{\mathcal{H}^\Gamma}(v)$.

- Property (T) inherits to finite-index subgroups and quotient groups.
- $\mathbb{Z}$ (or any infinite amenable group) does not have property (T).
  $\because \frac{1}{\sqrt{2k+1}} 1_{[-k,k]} \in \ell^2(\mathbb{Z})$ is asymp. $\mathbb{Z}$-invariant, but $\ell^2(\mathbb{Z})^{\mathbb{Z}} = \{0\}$.

$\rightsquigarrow$ Any f.i. subgroup of a property (T) group has finite abelianization.

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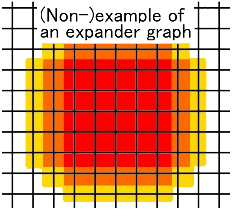
# An application of property (T): Expander graphs

**Definition**

A finite connected graph $X$ is an $\varepsilon$-**expander** if for $\forall A \subset X$ (vertices)

$$|\partial A| \geq \varepsilon |A|(1 - \tfrac{|A|}{|X|}).$$

- For $\mathcal{N}_k(A) := \{x \in X : \mathrm{d}(x, A) \leq k\}$, $|\mathcal{N}_k(A)| \geq (1 + \frac{\varepsilon}{2})^k |A|$ until it reaches $\frac{1}{2}|X|$. After that $|\mathcal{N}_k(A)^c|$ decreases by a factor $1 + \frac{\varepsilon}{2}$.
- Random walk on $X$ has mixing time $O(\log |X|)$.
- Want large $\varepsilon$-expanders with degree and $\varepsilon$ fixed.




**Explicit construction of expanders (Margulis 1973)**

$\Gamma = \langle S \rangle$ and $N \triangleleft \Gamma$ a finite index normal subgroup

$\rightsquigarrow X = \mathrm{Cayley}(\Gamma/N, S)$, where Edges $= \{\{x, xs\} : x \in \Gamma/N,\ s \in S\}$, is a $\kappa(\Gamma, S)^2$-expander.

E.g., $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, $S = \{I + E_{ij} : i \neq j\}$, and $X_q = \mathrm{SL}(3, \mathbb{Z}/q\mathbb{Z})$, $q \in \mathbb{N}$.

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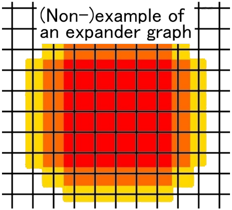


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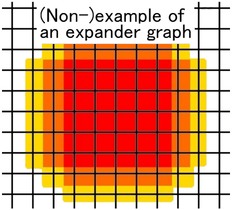


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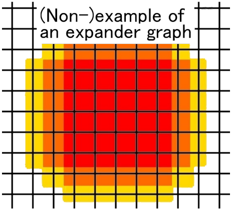


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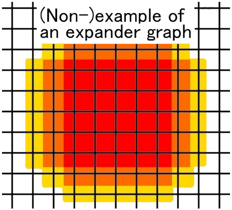


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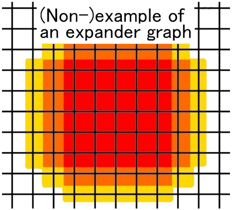


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# Some examples of property (T) groups

- $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, (Kazhdan 1967), but not $\mathrm{SL}_2(\mathbb{Z})$.
  - $\mathrm{EL}_n(\mathcal{R}) = \langle e_{ij}(r) : i \neq j, r \in \mathcal{R} \rangle \subset \mathrm{GL}_n(\mathcal{R})$, $n \geq 3$, where $\mathcal{R}$ finitely generated ring and $e_{ij}(r) := I_n + rE_{ij}$ (Shalom & Vaserstein, Ershov–Jaikin-Zapirain 2006–08).
  - $\mathrm{Aut}(\mathbf{F}_n)$, $n \geq 4$. (Kaluba–Nowak–O., K–Kielak–N., Nitsche 17–20).
    $\mathbf{F}_n \twoheadrightarrow \mathbb{Z}^n$ abelianization $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.
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  ⚠ The proof is heavily computer-assisted.

**Product Replacement Algorithm (Celler et al. 95, Lubotzky–Pak 01)**

$\mathrm{Aut}^+(\mathbf{F}_n) = \langle R_{i,j}, L_{i,j} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbf{F}_n)$, where $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$ and

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PRA is a practical algorithm to obtain "random" elements in a given finite group $\Lambda$ of rank $< n$ via the PRA random walk

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  - $\mathrm{EL}_n(\mathcal{R}) = \langle e_{ij}(r) : i \neq j, r \in \mathcal{R} \rangle \subset \mathrm{GL}_n(\mathcal{R})$, $n \geq 3$, where $\mathcal{R}$ finitely generated ring and $e_{ij}(r) := I_n + rE_{ij}$ (Shalom & Vaserstein, Ershov–Jaikin-Zapirain 2006–08).
  - $\mathrm{Aut}(\mathbf{F}_n)$, $n \geq 4$. (Kaluba–Nowak–O., K–Kielak–N., Nitsche 17–20).

    $\mathbf{F}_n \twoheadrightarrow \mathbb{Z}^n$ abelianization $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_n) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$.

    $\rightsquigarrow$ $\mathrm{Aut}(\mathbf{F}_2)$ does not have (T). Neither $\mathrm{Aut}(\mathbf{F}_3)$ (McCool 1989).

  ⚠ The proof is heavily computer-assisted.

**Product Replacement Algorithm (Celler et al. 95, Lubotzky–Pak 01)**

$\mathrm{Aut}^+(\mathbf{F}_n) = \langle R_{i,j}, L_{i,j} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbf{F}_n)$, where $\mathbf{F}_n = \langle g_1, \ldots, g_n \rangle$ and

$$R_{i,j} : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i g_j, g_{i+1}, \ldots, g_n),$$

$$L_{i,j} : (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_j g_i, g_{i+1} \ldots, g_n).$$

PRA is a practical algorithm to obtain "random" elements in a given finite group $\Lambda$ of rank $< n$ via the PRA random walk

$$\mathrm{Aut}^+(\mathbf{F}_n) \curvearrowright \{(h_1, \ldots, h_n) \in \Lambda^n : \Lambda = \langle h_1, \ldots, h_n \rangle\}.$$

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**Hilbert's 17th Pb**: $f \in \mathbb{R}(x_1, \dots, x_d)$, $f \geq 0$ on $\mathbb{R}^d$
(E. Artin 1927) $\Longrightarrow f = \sum_i g_i^2$ for some $g_1, \dots, g_k \in \mathbb{R}(x_1, \dots, x_d)$.

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$\Sigma^2\mathbb{R}[\Gamma] := \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$ **positive cone**

Here $\mathbb{M}_\Gamma^+$ finitely supported positive semidefinite matrices.

- $\mathbb{B}(\mathcal{H})^+ := \{A = A^* : \langle Av, v\rangle \geq 0 \ \forall v \in \mathcal{H}\} = \Sigma^2\mathbb{B}(\mathcal{H})$ psd operators.
- $\forall(\pi, \mathcal{H})$ unitary rep'n, $\pi(\sum_i f_i^* f_i) = \sum_i \pi(f_i)^* \pi(f_i) \geq 0$ in $\mathbb{B}(\mathcal{H})$.
- $C^*[\Gamma]$ the universal enveloping $C^*$-algebra of $\mathbb{R}[\Gamma]$.

**Laplacian:** For $\Gamma = \langle S\rangle$,

$$\Delta := \sum_{s\in S}(1-s)^*(1-s) = 2|S| - \sum_{s\in S}(s + s^{-1}) \in \Sigma^2\mathbb{R}[\Gamma].$$

Then, $\langle \pi(\Delta)v, v\rangle = \sum_{s\in S} \|v - \pi(s)v\|^2$ and

$\Gamma$ has (T) $\Longleftrightarrow \exists\lambda > 0 \quad \forall(\pi, \mathcal{H}) \quad \mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\lambda, \infty)$

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$\rightsquigarrow \kappa(\Gamma, S) \geq \sqrt{\lambda/|S|}$

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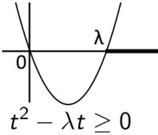

# Algebraic characterization of property (T)

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**Theorem (O 2013)**

$\Gamma$ has (T) $\iff \exists \lambda > 0$ such that $\Delta^2 - \lambda\Delta \succeq 0$ in $\mathbb{R}[\Gamma]$

**Stability** (Netzer–Thom): It suffices if $\exists \lambda > 0$ $\exists \Theta \in \Sigma^2\mathbb{R}[\Gamma]$ such that

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## Semidefinite Programming (SDP)

$\Gamma$ has (T) $\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$

$\iff \exists E \Subset \Gamma\ \exists\lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_E^+\}$

By fixing a finite subset $E \Subset \Gamma$, we arrive at the SDP:

`maximize` $\lambda$

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- Due to computer capacity limitation, we almost always take

$$E := \mathrm{Ball}(2) = \{e\} \cup S \cup S^2 = \mathrm{supp}\,\Delta \cup \mathrm{supp}\,\Delta^2.$$

$\rightsquigarrow$ Size of SDP: dimension $|E|^2$ and constraints $|E^{-1}E| = |\mathrm{Ball}(4)|$.

**Certification Procedure:**

Suppose $(\lambda_0, P_0)$ is a hypothetical solution obtained by a computer.
Find $P_0 \approx Q^{\mathrm{T}} Q$ (with $Q\mathbf{1} = 0$) and calculate **with guaranteed accuracy**

$$\|\Delta^2 - \lambda_0\Delta - \textstyle\sum_{x,y} (Q^{\mathrm{T}} Q)_{x,y} (1-x)^*(1-y)\|_1 \ll \lambda_0.$$

- Solving SDP is computationally hard, but certifying (T) is relatively easy.

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# Results

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**Results of SDP for $E = \mathrm{Ball}(2)$.**

- $SL_n(\mathbb{Z})$ with $S = \{e_{ij} : i \neq j\}$: $\lambda_3 > 0.27$, $\lambda_4 > 1.3$, $\lambda_5 > 2.6$. (Netzer–Thom 2014, Fujiwara–Kabaya 2017, Kaluba–Nowak 2017)
- No response for $SL_6(\mathbb{Z})$.

For $\mathrm{Aut}^+(\mathbf{F}_4)$, the size of SDP $\approx$ 10 000 000, beyond our computer's capacity. We exploited invariance under $\mathfrak{S}(n) \ltimes (\mathbb{Z}/2)^{\oplus n} \curvearrowright \mathrm{Aut}^+(\mathbf{F}_n)$.

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**Theorem**

$\mathrm{Aut}^+(\mathbf{F}_n)$ has property (T) for

- $n = 5$ (Kaluba–Nowak–O. 2017)
- $n \geq 6$ (Kaluba–Kielak–Nowak 2018, by “stability” explained below)
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**Results of SDP for $E = \mathrm{Ball}(2)$.**

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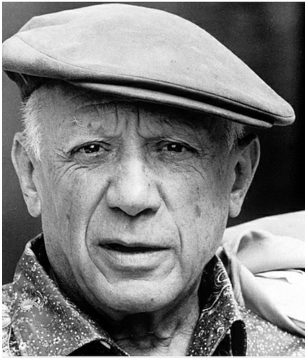

Revista Vea y Lea, January 1962

"But they (= computers) are useless.
They can only give you answers."
Pablo Picasso, 1968.

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# Property (T) for an infinite series (KKN 2018)

$\Gamma_n := \mathrm{Aut}^+(\mathbf{F}_n)$, $S_n := \{R_{i,j}, L_{i,j} : i \neq j\}$, $\mathrm{E}_n := \{\{i,j\} : i \neq j\}$

Want to show $\Delta_n = \sum_{s \in S_n} 1 - s$ satisfies $\Delta_n^2 - \lambda_n \Delta_n \succeq 0.$

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$$=: \mathbf{Sq}_n + \mathbf{Adj}_n + \mathbf{Op}_n .$$

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# Generalizing property (T) for $\mathrm{EL}_n(\mathcal{R})$ for a rng $\mathcal{R}$

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**Theorem (O. 2022)**

For any f.g. **comm. rng** $\mathcal{R}$ generated by $R_0 \in \mathcal{R}$ and for $n$ **large enough**,

$$\Delta := \sum_{r \in R_0} \sum_{i \neq j} (1 - e_{ij}(r))^*(1 - e_{ij}(r)) \quad \text{and}$$
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in $\mathbb{R}[\mathrm{EL}_n(\mathcal{R})]$ satisfy $\Delta^2 \geq \lambda \Delta^{(2)}$ in $\mathrm{C}^*[\mathrm{EL}_n(\mathcal{R})]$ for some $\lambda > 0$.

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