Conjugates, paths, and groups of DIFF’s

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Group (mathematics)

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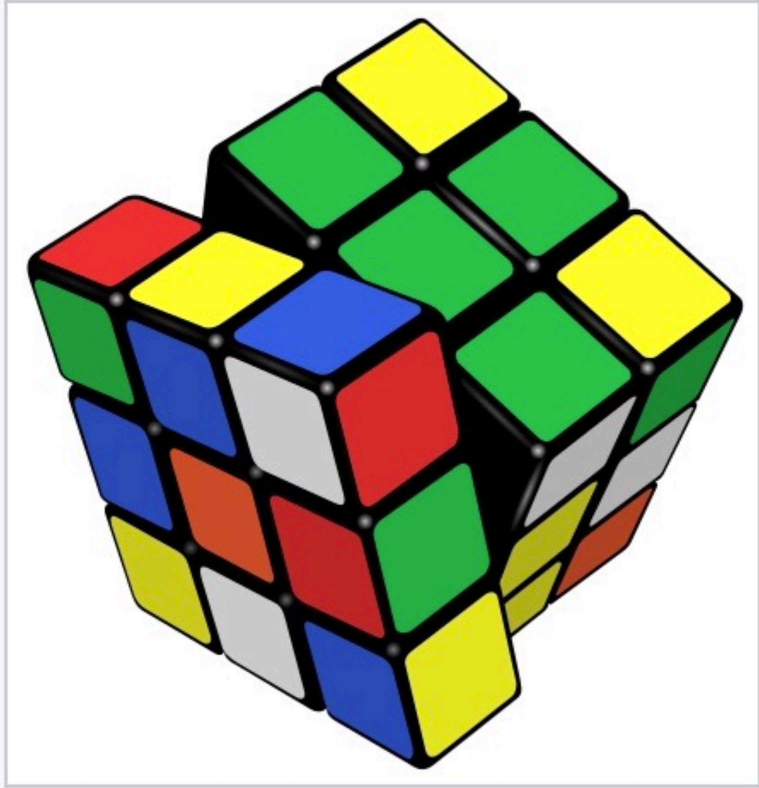
This article is about basic notions of groups in mathematics. For a more advanced treatment, see Group theory.

In mathematics, a **group** is a set and an operation that combines any two elements of the set to produce a third element of the set, in such a way that the operation is associative, an identity element exists and every element has an inverse. These three axioms hold for number systems and many other mathematical structures. For example, the integers together with the addition operation form a group. The concept of a group and the axioms that define it were elaborated for handling, in a unified way, essential structural properties of very different mathematical entities such as numbers, geometric shapes and polynomial roots. Because the concept of groups is ubiquitous in numerous areas both within and outside mathematics, some authors consider it as a central organizing principle of contemporary mathematics.[1][2]



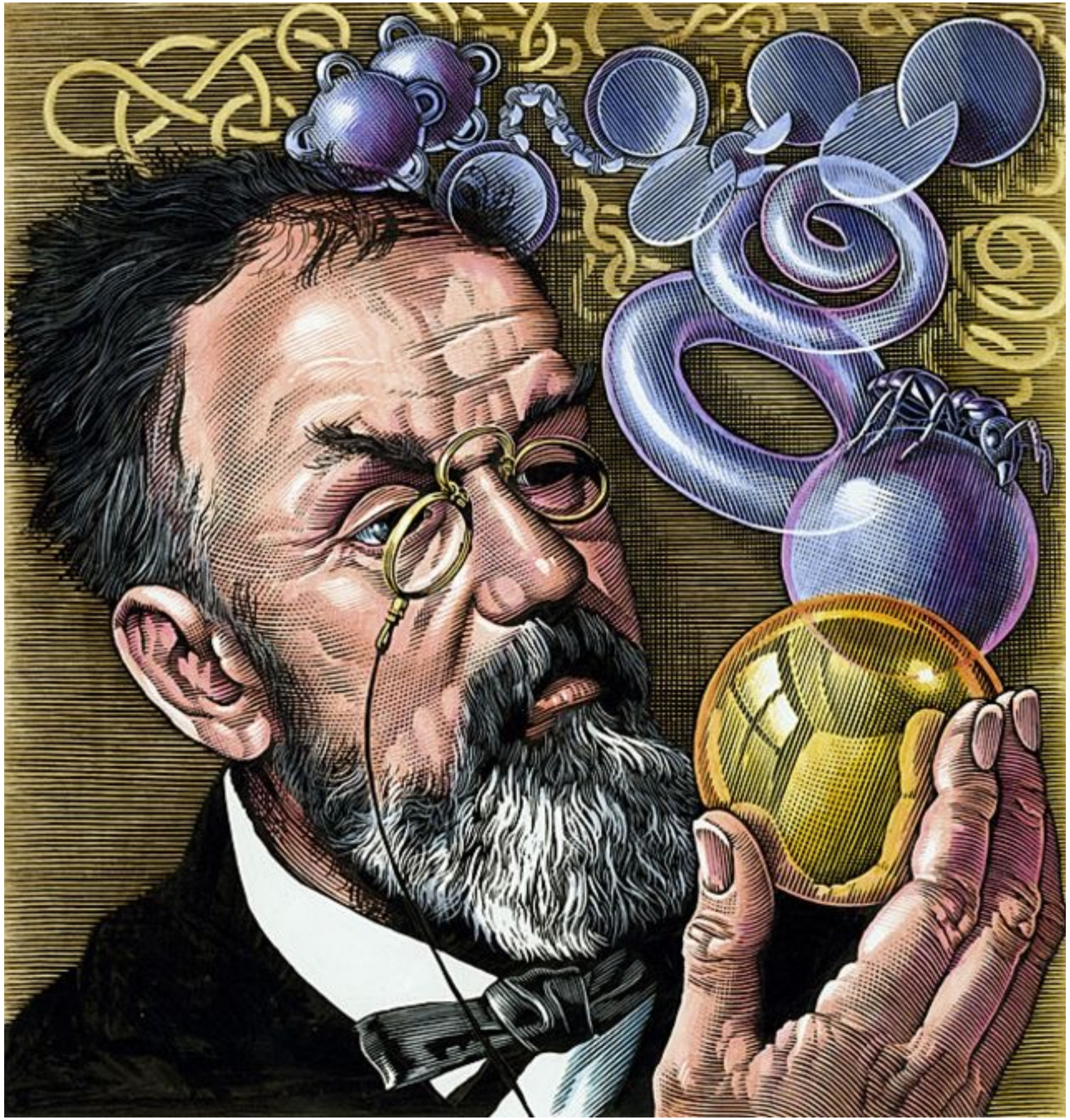
The manipulations of the Rubik's Cube form the Rubik's Cube group.

The genesis of groups

- Groups can be approached in many ways: pure algebraically (axiomatically), combinatorially, geometrically, **dynamically**,...

- Galois invented groups, but one can argue that it was Cayley (Dedekind) who opened the path to a wide understanding of groups...

“Les mathématiques ne sont qu’une histoire de groupes”
(H. Poincaré).



Évariste Galois
A tribute in numbers

Panmagic groups

The panmagic group is the group of permutations of the cells of a square that preserve the panmagic relations: they send panmagic squares into panmagic squares.

$$P_5 \sim (S_5 \times S_5) \rtimes \mathbb{Z}/2\mathbb{Z}$$

$$P_4 \sim (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$$

The algebraic theory of diabolic magic squares.

B. Rosser & R. Walker

Duke Math. J. 1939

Theorem
(Narayana Pandit,
B. Rosser & R. Walker,
H. Coxeter,
W. Müller, N.):

The 4x4 panmagic group has order 384, and is isomorphic to the group of symme-tries of the hypercube.

The Chautisa Yantra at Khajuraho
(centre of India)

IMAGES DES MATHÉMATIQUES

La recherche mathématique en mots et en images

- Des carrés magiques en cadeaux.
- Le carré magique de Khajuraho est un hypercube.

- If you get an idea for computing P_7 please tell me !
- In any case, you will lear how to build your own magic square...

Ramanujan's magic square

22	12	18	87
88	17	9	25
10	24	89	16
19	86	23	11

Groups as dynamical objects I

- Every group is a subgroup of the group of permutations of some object (Cayley).

Proof: The group acts on itself
(on its Cayley graph if finitely generated).

Groups as dynamical objects II

Every countable group can be realized as a group of homeomorphisms of the Cantor set:

G acts on $\{0,1\}^G$

Homeo (C) contains all countable groups !

Simple amenable groups

- Start with a homeomophism T of the Cantor set C.
- Then let G be the group of homeomorphisms of C obtained this way: for each element g of G there is a partition of C into finite-ly many clopen sets C_i so that the the restriction of g to each C_i is the restriction of some power (iterate) of T.

Theorem (Matui; Juschenko - Monod): If T is properly chosen, then [G,G] is finitely generated, simple and amenable.

From 0 to higher dimension

- Diff^r (M) keeps all the information about the mani-fold: it recognizes both M and *r* (Filipkiewicz; Mann, Hurtado, Kim - Koberda - de la Nuez González).

- The connected component of these groups are algebraically simple (Herman, Thurston, Mather).

  Remaining open case: $r = \dim(M) + 1$

Warning: $\text{Diff}^{1+bv}(S^1)$ is not simple (Mather).

Can these groups be distinguished "geometrically"

Definition: an (infinite order) element a in a finitely generated group is distorted if the powers a^n may be written as products of $o(n)$ number of generators (and inverses).

$$BS(1,2) = \langle a, b : aba^{-1} = b^2 \rangle$$

$$\longrightarrow \quad b^{2^n} = a^n b a^{-n}$$

Definition: an element of a general group is a distortion element if it is distorted inside some finitely generated sub-group (Gromov, Rosendal).

An obstruction and "two" specific examples

- If a diffeomorphism has a hyperbolic fixed (periodic) point, then it is undistorted in the group of C^1 diffeomorphisms.

- Every irrational rotation of the circle is distorted in the group of circle diffeomorphisms (Avila).

- Irrational rotations are also distorted in the group of piece- wise affine circle homeomorphisms (Banecki & Szarek).

Distorted diffeomorphisms

Problem: Given $r > s \geq 1$, give examples (or prove that there exist) C^r diffeomorphisms that are undistorted in Diff^r but distorted in Diff^s

Only one "relevant" example known: $r = 2$, $s = 1$, $M = S^1$, $[0,1]$.

(N; Dinamarca-Escayola).

Warning: These diffeomorphisms have no hyperbolicity behaviour (vanishing topological entropy).

Germs

- **G** = germs of analytic diffeomorphisms at the origin.
  Group operation: composition !

$$f : z \mapsto a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \ldots$$

Theorem (Cerveau, Cantat, Souto): This group contains the funda-mental groups of compact surfaces.

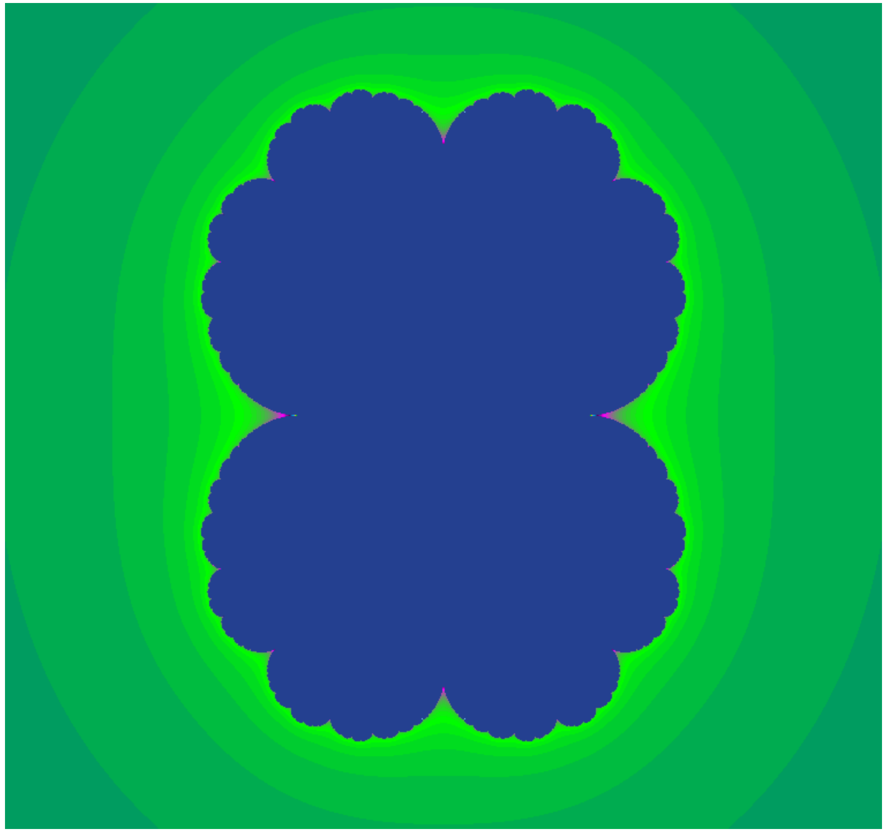
Producing a free subgroup inside G is already interesting...

Another challenging question

The two maps below generate a copy of BS (1,2) in G;
in particular, the second one is a distortion element of G.

$$a(z) = \frac{z}{2}$$

$$b(z) = \frac{z}{1+z} = z - z^2 + z^3 - z^4 - \dots$$



Question: Is $c(z) = z - z^2$ a distortion element ?

Three derivatives, and more...

$$L_{fg}(x) = L_g(x) + L_f(g(x)), \qquad L_h(x) := \log(Dh(x))$$

$$A_{fg}(x) = A_g(x) + A_f(g(x)) \cdot Dg(x), \qquad A_h(x) := \frac{D^2h(x)}{Dh(x)} = D(\log Dh(x))$$

$$S_{fg}(x) = S_g(x) + S_f(g(x)) \cdot (Dg(x))^2, \qquad S_h(x) := \frac{D^3h(x)}{Dh(x)} - \frac{3}{2}\left(\frac{D^2h(x)}{Dh(x)}\right)^2$$

A nice formula for the Schwarzian derivative

$$S_g(x) = \frac{1}{6} \lim_{y \to x} \left[\frac{Dg(x)\, Dg(y)}{(g(x) - g(y))^2} - \frac{1}{(x - y)^2} \right]$$

Thurston's stability theorem

- **Thurston's trick**: If G is a finitely generated group of germs of C^1 diffeomorphisms at the origin, then there exists a nontrivial homomorphism $G \to R$.

"Proof": $g \to L_g(0) = \log(Dg(0))$

But... what happens if this homomorphism is trivial ?

If more derivatives are available...

$$g(x) = x + a_i x^i + a_{i+1} x^{i+1} + \dots$$

Two group homomorphisms:

$$g \mapsto a_i$$

$$g \mapsto \frac{i\,a_i^2}{2} - a_{2i-1}$$

The spirit of the proof

$$E(x) = \pm e^{-1/x^2}$$

- **Exercise :** The conjugate by E^{-2} of every germ of C^k diffeo-morphism at the origin is the germ of a C^k diffeomorphism that is tangent to the identity up to order k (Muller-Tsuboi).

- **Thurston's idea:** "Renormalize" the geometry near the origin in order to create something like a nontrivial (logarithmic) derivative

Some consequences

- Thurston's stability provides many examples of groups that, though acting by homeomorphisms on an interval, do not act by diffeomorphisms.

Example: braid groups B_n , n > 4

(Dehornoy; Dehornoy-Rolfsen-Wiest; Nielsen-Thurston).

- **This is not the only obstruction !**

(Calegary, N, Bonatti-Monteverde-N-Rivas, ...)

$$G := F_2 \ltimes \mathbb{Z}^2 \subset \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$$

Property (T) also gives an obstruction

- A group G has property (T) if every action by isometries on a Hilbert space has a fixed point (Serre).

- Can reformulated in a cohomological language: every cocycle with respect to an unitary representation is a coboundary.

Theorem (N): Every f. g. group of $C^{3/2}$ diffeomorphisms of the circle (resp. interval) is finite (resp. trivial).

A tool for the proof:

$$c(g)(x,y) := \frac{\sqrt{Dg(x)\, Dg(y)}}{g(x)-g(y)} - \frac{1}{x-y}$$

Key point: If *g* is a diffeomorphism of class $C^{3/2}$,

then c(g) belongs to $L^2(\mathrm{S}^1 \times \mathrm{S}^1)$

(although (x,y) → 1/(x-y) does not !)

Life is not always smooth

- **Question:** Does there exist an infinite group of circle homeo-morphisms with property (T) ?

Negative direction: Witte-Morris, Ghys, Deroin-Hurtado

Positive direction: Ozawa ...

- **Question:** Does there exist an finite group with property (T) which is left orderable ?

Obstructions arise in any regularity

- **Theorem (Kim - Koberda; Mann - Wolf):** For every $r > s \geq 1$, there exists a finitely generated group of C^s diffeomorphisms of the interval / circle that does not embed into Diff^r

- Very simple general questions remain open in higher dimension:

Question: Does there exists a f. g. torsion-free group that does NOT embed into the group of homeomorphisms of the plane ?

Candidates / tools: Monsters ? (Tarski, Osin, ...)

- Very important recent achievements; e.g. solution of the Zimmer conjecture by Brown-Fisher-Hurtado (lattices do not act unnaturally...). Related to classical results of Mostow, Margulis, ...

Groups that do act !

- Take two maps "at random". Then they will generate a free group.

Warning: No two piecewise affine homeomorphisms of the interval will generate a free group (Brin-Squier).

- Take a diffeomorphism "at random". Then the set (group) of diffeomorphism that commute with it is reduces to its powers.

"A generic diffeomorphism has a trivial centralizer"

(Smale, Kopell; Palis, Yoccoz, ... ; Bonatti-Crovisier-Wilkinson)

In general, diffeomorphisms do not arise from vector fields (Palis, Milnor...)

Spaces of diffeomorphisms

- Studing the space of diffeomorphisms of a manifold is already difficult (homotopy type, etc).

Example (Alexander): The space of homeomorphisms of a ball is contractible.

Theorem (Smale): The space of diffeomorphisms of the sphere has the homotopy type of SO(3).

Spaces of commuting diffeomorphisms

- Almost nothing is known, even in dimension 1 !

Question (Rosenberg): Is it locally path-connected for S^1 ?
This was (indirectly) treated by Yoccoz in his thesis...

Theorem (Hélène Eynard-Bontemps - N): The space of C^{1+ac} commuting diffeomorphisms of the circle is path connected.

- Also true in class C^1 but much easier (and less interesting...).

Two naive approaches

- Alexander trick works well at least for commuting homeomor-phisms of the interval, but it does not preserve regularity...

- A homeomorphism of the interval can be sent by a linear ho-motopy (of its graph) to the trivial one, but this procedure does not preserve commutativity...

Several pitfalls along the path

- If f is a C^2 circle diffeomorphism with no periodic point, then it is conjugated to a rotation (Denjoy).
- The same holds for commuting diffeomorphisms (simultaneous conjugacy).

Pitfall: in general, the conjugacy is not smooth (Arnold, Herman, Yoccoz, Moser, Pérez-Marco, Fayad-Teplinsky, ...)

Another pitfall

- If f is a C^2 diffeomorphism of the interval [0,1) with no fixed point at the interior, then it is the time-1 of a flow (Szekeres). Work of Kopell clarifies the situation for commuting maps.

Pitfall: In general, the flow is not more regular than C^1 (Sergeraert).

- It may happen that such a diffeomorphism has no C^2 square root...
- It may happen that the C^2 maps of the flow are those that arise from combinations of two irrationally independent numbers (E-B).

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ $\to$ $\varphi_{X_0}^t = \varphi_{X_1}^t \quad \forall t \in \mathbb{Z} + \alpha\mathbb{Z}$

$X_0 = X_1$ on $(0,1)$

$\rightsquigarrow X$ C^1 on $[0,1]$ — 2nd case.

Sergeraert constructed a C^1 X s.t. φ_X^n are C^∞

φ_X^t

Still another pitfall

- If f is a C^2 diffeomorphism of $[0,1]$, then there are two (Szekeres) vector fields (left and right).

Pitfall: these may be different (Mather, Yoccoz).

But, some structure arises. For instance, if these vector fields are different then the C^1 centralizer is infinite cyclic (Mather, Yoccoz).

A key idea: paths of conjugates

- Inspiration: a parabolic element in Mob is conjugate to its roots...

- Try to conjugate a group action so that it becomes closer and closer to the trivial action (or, at least, to an action by rotations).

In C^0 topology: given $f\,g = g\,f$, we want h_t such that

$$h_t\, f\, h_t^{-1} \to R_1 \qquad h_t\, f\, h_t^{-1} \to R_2$$

Always possible !

A first obstruction

- Hyperbolic fixed points are invariant under C^1 conjugacy...

- **Theorem (N) :** Hyperbolic periodic points are the only obstruc- tion to conjugate a circle diffeomorphism to diffeomorphisms close to rotations.

Key observation: Hyperbolic periodic points are detected by the growth of the logarithmic derivative: $L(g^n)$

A second obstruction

- In class C^{1+ac} , the problem is related to the growth of the affine derivative:

$$A(g^n) = D^2(g^n)/D(g^n) = D(\log(Dg^n))$$

$$\|A(g^n)\|_{L^1} = \int \left|D(\log(Dg^n))\right| = \mathrm{var}(\log Dg^n)$$

A structure result

- **Theorem (N, Eynard Bontemps-N):** The growth of the affine derivative is linear only in two cases:

  - presence of hyperbolic periodic points;

  - dynamics on the interval for which left and right vector fields do not coincide (here, Mather's theory applies...)

- In case of sublinear growth, the affine derivative is "almost a coboundary", which means that the action can be conjugated to actions closer and closer to rotation actions...

Summary:

- We do affine interpolation but not of the graphs...
- We interpolte derivatives !!!
- Since these are cocycles, we can keep the commutativity relation along the path.
- We detect the possible obstructions: these are related to the growth of the derivative cocycles.
- We use / establish theorems for the cases where these obstructions actually appear.
- With some extra (somewhat painfull) work, this gives a proof...

A last pitfall

Very surprisingly, hyperbolic fixed points are not so easy to handly because... they may be non linearizable !

Theorem (Eynard-Bontemps - N): there exist plenty of non-linearizable C^1 vector fields...

A last exercise

$$g_t(x) = e^{-t} \cdot x, \qquad h(x) = x\,|\log(x)|$$

$$f_t(x) = (h^{-1} g_t\, h)(x)$$

$\boldsymbol{f_t}$ is a flow of C^{1+ac} diffeomorphisms with the same multiplier as e^{-t} yet the conjugacy h is not bilipschitz...

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